GAUDI'S IDEAS FOR YOUR CLASSROOM: GEOMETRY FOR THREE-DIMENSIONAL CITIZENS

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How Johnny became a flatlander

This is the sad story of Johnny, a boy who was born in Sphereland (Burger, 1965), a place with three dimensions but became, in a few short years, a pure flatlander (Abbot, 1880).

From the very beginning, Johnny was fascinated by the shapes which formed his close environment. After spending quite a lot of time having fun in his oscillating cradle, he started to move all around, crawling upon all-fours. In this way, he became familiar with the corners and lower parts of the furniture, the texture of the floor and the intriguing shapes and colours of his toys. His beautiful tumbling-cap provided him with some protection against treacherous edges.

Afterwards, Johnny started walking and talking with increasing confidence, dancing through gardens and rooms, and discovering all kinds of directions (front, back, below, above...) and transformations (dancing, looking into mirrors, falling...). In his early kindergarten activities Johnny enlarged his three dimensional experience, but as soon as he entered school things changed dramatically. Movement was replaced by compulsory sitting, the games corner was replaced by tables, toys disappeared and work with pencils on flat white sheets took over. Since those days, Johnny's education focused on his mastery of the white flatland through writing, reading, counting, and drawing... and as soon as Johnny became familiar with computers and television sets, he realised that these intriguing all-moving-colour screens were also... essentially flat. One day, Johnny became especially angry because he did not do very well in a set of psychological tests given in class. These tests, which consisted of exercises based on sketches of cubes, proved that his spatial education was quite limited.

Several years of compulsory mathematics did not help to improve Johnny's perception of space in any great way. He realised that for some unknown reason, almost all teachers had no time to explore space, but almost all of them promised their young pupils that when they became adults, they would be prepared to visit space with the plane-culture they had acquired. Johnny thought that perhaps knowledge of space had something to do with knowledge of sex, since the right time would always be later and never now. But one day at the School the teacher said "Today we will study the space". It was like a dream but, immediately, the teacher added

- "Write, please. The space is R^3, i.e., the ternary cartesian product of the real line".
- "Oh! My God" said Johnny, who asked:
- How will we become familiar with the space? Will we be involved in experimental work?

- No! -replied the teacher- We will know the space by dealing with vectors and inner products...

Later Johnny went to a University as a math major and had opportunities to discover that R^3 was a particular case of a linear vector space over a field. But time passed, and Johnny's education was completed and he went to work and became a fine citizen and a mathematics teacher of the universal Flatland.

On Gaudi's legacy

Antoni Gaudí (1852-1926) was an exceptional architect who succeeded in combining imagination, ingenuity and creativity in all his masterpieces. His celebrated old buildings, mainly located in the Barcelona area (Spain), and especially his Temple of the Sagrada Familia (Holy Family), which is still under construction, show how he managed to combine the geometry of shapes with stable structural design and great originality, in keeping with Vitruvius' classical claim "Architecture is form, function and beauty".

Gaudí received a long training as an architect and civil engineer with a solid grounding in mathematics. As a student he was mediocre, but as a practising professional he became a master. The key to this transformation was his ability to construct all he needed by himself, by combining his enormous spatial intuition, constructional criteria and experimental work.

Gaudí worked on his own spatial insight, by observing, thinking spatially...

> *"In books you don't find what you are looking for and, if you do find it, frequently it's wrong, so at the end you need to think directly..."*

by experimenting with objects, he developed a creative intuition which was ready for application...

> *"My structural and aesthetic ideas have a logic which cannot be denied. I have been thinking a lot why these ideas have not been applied before and this provokes my doubts. But since I am so convinced of their perfection, I have the obligation to apply them..."*

He rejected analytical approaches, and always thought directly in three dimensions:

> *"Geometry simplifies constructions, algebraic expressions complicate them".*

...but Nature and existing realities served him as a reference:

> *"This tree, near my workshop, is my master".*

After imagining a project, he paid great attention to all possible details:

> *"I carry out computations and experiments, I pay attention to all the details... thus the logical shape is born from needs".*

In this way, he searched for a fine equilibrium between rational and emotional characteristics:

> *"For a work to be beautiful, all its elements must be in the right location, must have the right dimension, the right shape, the right colour... to obtain harmony, you need contrast".*

Some of his geometrical findings [see e.g., (Gomez, 1996), (Collins, 1960), (Tokutochi, 1983)] include the original use of ruled surfaces in Architecture; the inclination of columns, the catenary arches, the fractality in structures, etc.

He was the first architect to realise that the easiest arch to build is the one that handles its own weight, i.e., the catenary form. So these arches were made by symmetrisation of the form of a chain hanging between two points. The Sagrada Familia is the project where all quadratic rule surfaces (quadrics) are used.

For all of us Gaudí's legacy is not only a set of historical sites but also an interesting collection of ideas. Gaudí's ideas can guide us today as we face the challenges of educating three-dimensional citizens.

Note. You may visit the new web http://www.gaudi2002.bcn.es as well as the main webs linked to this one. Recent books in English about Gaudi are quoted at the end of the bibliography.

Space is the answer; what is the question?

Spatial questions have received a great deal of attention in the mathematics education literature and all of us benefit today from many research sources (Piaget, Vinh-Bang, Inhelder, Lunzer, Lovell, Ogilvie, Laurendan, Pinard, Carpenter, Gaulin, Hart, Vergnand, Freudenthal, Gattegno...). Plans for official curricula are paying more and more attention to geometrical education (see e.g. (NCTM, 2000)). Evaluation programs assess spatial skills (see, e.g., (De Lange, 1999)) but in many places teachers are not confident about dealing with three-dimensional geometry, there is a lack of good three-dimensional models in the teaching resource catalogues, and what is worse, most children end compulsory schooling without spatial literacy.

We have carried out several educational projects over recent years involving spatial geometry workshops with children of 12-14 years old (*Matemàtica a Torrebonica*), with high-school pupils of 17-18 years old (*Trobades amb la Ciencia*) with maths majors and architectural students (FME-ETSAB, UPC) and with secondary school teachers (OMA, Argentina). All these experiences proved rewarding for most of the participants, and from them we have developed some ideas for improving spatial literacy that we would like to share.

Towards a spatial culture

Below there are 2^3 principles that could guide us, as mathematics educators, in providing our students with a spatial culture.

1. **Visual thinking in three dimensions, which is a key point in spatial culture, must not be confined to early grades and can be stimulated at all levels.**

Many teachers believe that making models and experimenting with 3D may have a role in early grades but "it is something to be replaced by more sophisticated linguistic and symbolic descriptions", i.e., "real mathematics comes after experimental work". This belief is wrong and research has shown that if you do not provide a stimulating reference for abstract concepts then formal approaches degenerate into a mere intellectual game. Visual thinking is not just an appetiser for the main course of abstraction. Clearly, at different levels one is restricted to some spatial items but there are opportunities to offer a broader spatial culture at all ages. Since *visual thinking* is a key word in spatial culture, perhaps it is useful (Senechal, 1991) to clarify the concepts of visualisation and visual thinking.

Visualisation is any process producing images (1D, 2D, 3D, graphs, diagrams...) so visualisation is a useful tool for visual thinking but this way of thinking goes far beyond visualisation techniques. In this regard, space representation and space perception (as mental reconstruction of representations of three-dimensional objects) are important components of spatial visual thinking. This makes polyhedra a useful topic. Following Rudolf Arnheim, visual thinking can be:

> *"...active exploration, selection, grasping of essentials, simplification, abstraction, analysis and synthesis, completion, correction, comparison, problem solving, as well as combining, separating, putting in context..."*

Example 1. *What is the maximum number of faces of a cube that you can see?*

The answer is 6... but you must be small and sitting inside the corner of the cube. The answer is 3 if you are looking at a small cube in your hand and you are an outside observer!

2. Spatial common sense must be cultivated and developed since it is not necessarily an innate capacity

There is much discussion about common sense and mathematics education (see, e.g. (Howson, 1998), (Keitel, 1996)). Here, by spatial common sense I mean development of the capacity to deal with three-dimensional questions using the most natural and intuitive approaches to the problem.

This is not a "spatial awareness" or a culturally-dependent "average understanding". If spatial common sense is not properly cultivated and worked on in maths activities, then students may have difficulties in problem solving when facing 3D objects or transformations. In particular, if this sense is not developed, students try to apply sophisticated, unnatural solutions to space questions which often have answers which are quite obvious.

Example 2. *You have a closed box and a graduated rule. How can you evaluate its interior diagonal?*

Most people measure outside diagonals and sides of the box and by virtue of Pythagoras' theorem they compute the value of the diagonal. Spatial common sense tells you to locate the box in the corner of the table and to move (measuring) the box in such a way that you can measure directly the diagonal of the exact empty space left by the box (distance from the corner of the table to the corresponding corner of the box).

3. Spatial culture requires: a break with the traditional educational ordering of dimensions (line-plane-space) and a reordering of some technical difficulties.

There is a general feeling among teachers that "mathematical difficulties necessarily require a curricular ordering of concepts". This implies, for example, not dealing with spatial figures until study of the plane has been completed, not presenting certain shapes because their equations are of a higher degree, not presenting surfaces before having studied curves, etc. The result is generally very poor familiarity with three-dimensional mathematics. My claim is that by finding "appropriate descriptions" one can explore space without using technical difficulties as a criteria for selection. Intuition may allow one to face realities that the majority of citizens would never be able to see if advanced mathematical knowledge was required.

Example 3. *How can you make a model of a cylinder which exhibits an elliptic section?*

This seems like an advanced level question. But it is easy to discover that with the graph of any sinusoidal curve we can make a model of a piece of cylinder showing on elliptic section.

4. Spatial culture can be based upon the reality of our world, exploring its many interesting faces and solving real world problems.

Our reality is rooted in the blue planet and progress of mankind. Our main mathematical laboratory is this reality. Trees, mountains, stones, animals, flowers, roads, buildings, tools... constitute the 1:1 models for exploration, representation and transformation. If we are sensitive to this real context (Lange, 1998) we can easily motivate the learning process and use real things themselves as teaching materials.

Example 4. *How can you pinpoint the level of liquid in a glass if you just want a half glass?*

If the glass is a cylinder then "half height=half volume". What happens if the glass is conical in shape? Half height would imply half diameter of the liquid upper surface... so 1/8 of the total volume. If you want ½ volume you need to multiply the height by $\sqrt[3]{\frac{1}{2}}$, i.e., to take 80% of it. It is remarkable that $\sqrt[3]{2}$ plays a role in drinking (or in ice cream orders!).

5. Spatial culture may be enriched by means of appropriate use of different languages, technologies and mathematical models.

Spatial culture cannot be exclusively restricted to the development of a given mathematical model (Euclidean geometry, non-Euclidean geometry...). This has always been a major problem in the teaching of geometry, in which misguided debates on models and languages have sometimes led to the death of geometry education. I would argue that by combining different languages (synthetic descriptions, symbolic representations, graphical work, etc.) and different geometrical mechanisms (coordinates, vectors, parameters, formulas...) it is possible to improve spatial culture. Applying the appropriate tools to each problem is far more instructive than forcing artificial use of inappropriate tools.

Example 5. *Mark a helix with a slope of 10% on a cylinder surface.*

Let h be the measured height of the cylinder. Construct a right-angle triangle with legs h and 10h... The slope of the hypotenuse will be $100 \cdot {}^{h}\!/_{10h}$ =10% and you just need to wrap the triangle round the cylinder.

6. Spatial culture can include artistic dimensions, and is therefore an ideal area for interdisciplinary links

While our main interest is to focus on the geometrical understanding of space, it is also true that an exploration of space provides good opportunities to show links to natural sciences (deserts, forests, rivers, mountains...), the history of mankind's knowledge of space, artistic considerations (sculpture, perspective, design...), architectural solutions (buildings, structures...) technological impact (robots, machines, virtual reality...), etc. For these items see, e.g. (Bolt, 1991), (Cook, 1979), (Hargittai, 1986), (Malkevitch, 1991), McMahon, 1983), (Salvadori, 1990), (Senechal-Fleck, 1988), (Steen, 1984) among others.

Example 6. *Making 3D-kites and cutting diamonds have a geometrical shape in common: the octahedron.*

You can buy or construct a 3D-kite based on the regular octahedron and enjoy flying it (on a windy day) while paying attention to the physical need for it to have some flexibility in the structure. If you note also the parallelism of some faces of the octahedron you may discover how to make a beautiful table and you may have a fascinating journey into the world of jewels when discovering how the octahedron is the basic shape for diamond cutting.

7. Spatial culture provides opportunities to promote the research spirit in mathematics classrooms

While the plane offers many well-known mathematical objects, space offers intriguing open-ended questions that can be best faced in a spirit of research. We need to promote the idea that mathematics is a living field with interesting problems to be posed and/or solved. Space facilitates that approach. The following example is a recent one (Alsina, Garcia-Roig, 2000).

Example 7. *The golden number 1.618... is flat. The plastic number 1.314... is three-dimensional.*

The golden number $\phi = (1+\sqrt{5})/2$ is obtained as the limit of the ratios 1/1, 2/1, 3/2, 5/3... of the Fibonacci sequence and rectangles with proportion $a/b=\phi$ have the characterisation exhibited in Figure 1. The three-dimensional analogy for boxes is shown in Figure 2.

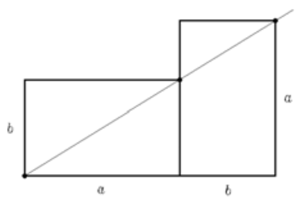

Figure 1

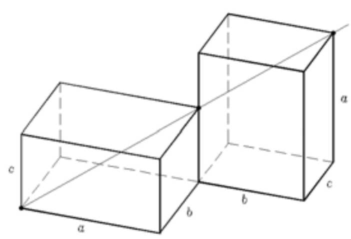

Figure 2

A box with sides a>b>c satisfies this spatial relation if and only if $a=P^2c$, $b=Pc$ and c is arbitrary, where P is van der Laand's plastic number 1.314... Note that P is the limit of the ratios of the sequence 1, 1, 1, 2, 2, 3, 4, 5, 7, 9... and P is the only positive real solution of the equation $x^3=1+x$. Do you remember that ϕ is the solution of $x^2=1+x$?... Clearly ϕ is attached to dimension 2.

8. **Spatial culture can provide future citizens with tools for developing basic spatial skills and can open a window to creativity.**

This factor is especially important. On completion of compulsory education, students should be able to successfully face basic spatial tasks in their daily lives as citizens (driving, buying furniture, cooking, decorating, reading maps, drawing, packing...), but it would be marvellous if they could also be imaginative, and creative in their jobs and lives.

> **Example 8.** *A person has long hair and decides to bunch all his/her hair together at a point above his/her head and cut it with scissors at that point. Is this a good option*?
>
> All the hair will be shorter... but of different length. Indeed the hair will form (if the person is suspended by his/her feet) a half hyperboloid of two sheets. So be careful with the spatial culture of your hairdresser!

Time for action

Motivated by their needs, people may find three-dimensional solutions by themselves (as Gaudí did). But our aim as educators is to improve spatial culture for all. Geniuses and people with practical-intuitive talents will always exist, independently of our educational efforts. This is not the problem. Our responsibility is to enable the general students in our classrooms to learn about and enjoy space.

On one occasion, Gaudí showed one of his visitors a range of geometrical models and, after describing their secrets and uses with emphatic use of his hands, he remarked, with excitement in his eyes*:*

“Wouldn't it be beautiful to learn geometry in this way?”

This is our challenge.

Three dimensions... or more?

But can we restrict our job in the classrooms just to the 3 spatial dimensions? What about time? What about colour, temperature,... feelings, emotions... *The "3 dimensions problem" is, in some sense, a metaphor. We deal, mainly, with 3 non-measurable dimensions: the interest for people, the interest for maths... and the interest for education.* The result of this components is... **the strength to love**. Thank you for making all this possible.

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